Section Two: Calculator-assumed 65% (97 Marks)

This section has **thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 (6 marks)

A system of equations is shown below.

$$
\begin{aligned}
x + 2y + 3z &= 1 \\
y + 3z &= -1 \\
-y + (a^2 - 4)z &= a + 2
\end{aligned}
$$

(a) Determine the unique solution to the system when $a = 2$. (2 marks)

$x + 2y + 3z = 1$

$y + 3z = -1$

$-y = 4$

$\therefore$ $y = -4$ ✓

$z = 1$

$x = 6$ ✓

(b) Determine the value(s) of a so that the system

(i) has an infinite number of solutions. (3 marks)

$R_2 + R_3$ $\quad 0 \quad 0 \quad a^2 - 1 \;\vdots\; a + 1$ ✓

infinite if $a^2 - 1 = 0$ and $a + 1 = 0$ ✓

$\Rightarrow a = -1$ ✓

(ii) has no solutions. (1 mark)

$a = +1$ ✓

Question 10 **(8 marks)**

The length of time, T months, that an athlete stays in an elite squad can be modelled by a normal distribution with population mean μ and population variance $\sigma^2 = 15$.

(a) An independent sample of five values of T is $7.7, 15.2, 3.9, 13.4$ and 11.8 months.

(i) Calculate the mean of this sample and state the distribution that a large number of such samples is expected to follow. (2 marks)

$$\frac{52.0}{5} = 10.4$$

Normal (by CLT), $\bar{X} = 10.4$, $s_{\bar{x}} = \frac{\sqrt{15}}{\sqrt{5}} = 1.732$

(ii) Use this sample to construct a 90% confidence interval for μ, giving the bounds of the interval to two decimal places. (3 marks)

$$10.4 \pm 1.645 \times \sqrt{3}$$

$$\Rightarrow \quad 7.55 \leq \mu \leq 13.25$$

(b) Determine the smallest number of values of T that would be required in a sample for the total width of a 95% confidence interval for μ to be less than 3 months. (3 marks)

$z = 1.96$ tolerance 1.5

Solve $\dfrac{1.96 \times \sqrt{15}}{\sqrt{n}} = 1.5$

$n = 25.61$

ie $n = 26$ is smallest such sample

Question 11 **(7 marks)**

Plane p_1 has equation $3x + y + z = 6$ and line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$.

(a) Show that the line l lies in the plane p_1. (3 marks)

Substitute $r = \begin{bmatrix} 1+t \\ 1-2t \\ 2-t \end{bmatrix}$ ✓ into P_1 which is $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 6$ ✓

$\begin{bmatrix} 1+t \\ 1-2t \\ 2-t \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 3+3t+1-2t+2-t = 6$ ✓

i.e. r lies in P_1

or $P_1 \perp \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ r is $\parallel \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ ✓ $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = 0$ ✓

and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ satisfies $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 6$ ✓ $\therefore r$ lies in P_1 (point + parallel)

(b) Another plane, p_2, is perpendicular to plane p_1, parallel to the line l and contains the point with position vector $i - 3j - k$. Determine the equation of plane p_2, giving your answer in the form $ax + by + cz = d$. (4 marks)

$\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ ✓ $\times$ ✓ $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$ ✓

$\therefore P_2$ is $x + 4y - 7z = -4$ ✓ at $(1, -3, -1)$

Question 12 **(11 marks)**

An object, initially at rest, is dropped from the top of tall building so that after t seconds it has velocity v metres per second.

The air resistance encountered by the object is proportional to its velocity, so that the velocity satisfies the equation $\frac{dv}{dt} = 10 - kv$, where k is a constant.

(a) Express the velocity of the object in terms of t and k. (~~4~~ 5 marks)

$$\int \frac{dv}{10-kv} = \int dt \quad \checkmark \quad \text{or} \quad \text{dsolve}(y' = 10 - ky, x, y) \checkmark$$

$$\therefore \frac{-\ln|10-kv|}{k} = t + c \quad \checkmark \qquad y = \frac{10 - e^{-kx+c}}{k} \quad \checkmark$$

$$y = v = 0 \text{ at } t = x = 0$$

$$\ln|10-kv| = -kt + c_1$$

$$10 - kv = e^{-kt} \cdot C_2$$

$$kv = 10 - C_2 e^{-kt}$$

$$v = \frac{10}{k} - \frac{C_2 e^{-kt}}{k} \quad \checkmark\checkmark$$

$$(0, 0) \Rightarrow C_2 = 10$$

$$\therefore v = \frac{10}{k}\left(1 - e^{-kt}\right) \quad \checkmark \text{ or similar}$$

(b) Sensors on the object indicate that its velocity will never exceed 55 metres per second. Determine the value of the constant k. (1 mark)

$$\frac{10}{k} = 55$$

$$k = \frac{10}{55} = \frac{2}{11} \quad \checkmark$$

(c) Another particle is moving along the curve given by $y = \sqrt[3]{x}$, with one unit on both axes equal to one centimetre. When $x = 1$, the y-coordinate of the position of the particle is increasing at the rate of 2 centimetres per second.

(i) Show that the x-coordinate is increasing at 6 centimetres per second at this instant. (2 marks)

At $x = 1 \quad \frac{dy}{dt} = 2$

$y = x^{\frac{1}{3}} \Rightarrow \frac{dy}{dt} = \frac{1}{3x^{\frac{2}{3}}} \cdot \frac{dx}{dt}$ ✓

$\therefore \frac{dx}{dt} = 2 \times 3 \times 1 = 6$ cm/sec ✓

(ii) Determine the exact rate at which the distance of the particle from the origin is changing at this instant. (~~4~~ marks) 3

$d^2 = x^2 + y^2$ ✓

$2d \frac{dd}{dt} = 2x \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$ ✓

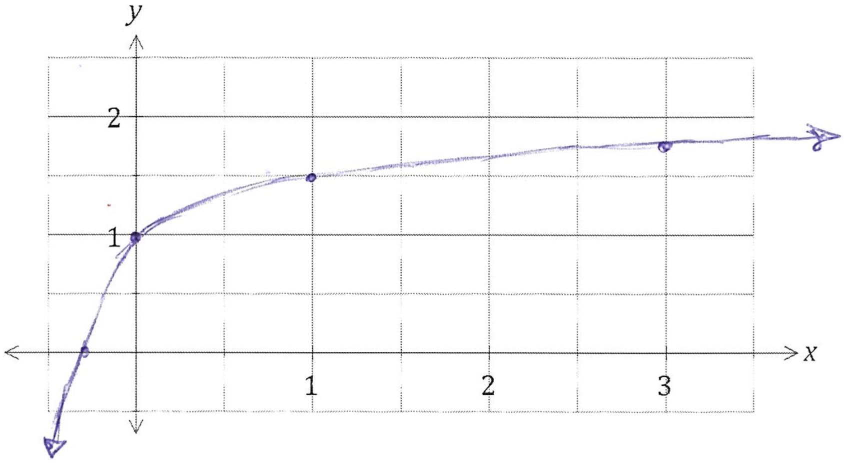
at $x = 1, \ y = 1, \ d = \sqrt{2}$

$\therefore \frac{dd}{dt} = \frac{1}{\sqrt{2}}(1.6 + 1.2)$

$= 4\sqrt{2}$ cm/sec ✓

Question 13 **(7 marks)**

(a) Sketch the graph of $y = \frac{2x+1}{x+1}$ on the axes below. (2 marks)



✓ $(0,1)$

✓ smooth shape + asymptote to 2

Simpson's rule is a formula used for numerical integration, the numerical approximation of definite integrals. When an interval $[a_0,\ a_n]$ is divided into an <u>even</u> number, n, of smaller intervals of equal width w, the bounds of these smaller intervals are denoted $a_0,\ a_1,\ a_2, \dots,\ a_{n-1},\ a_n$. Simpson's rule is:

$$\int_{a_0}^{a_n} f(x)\,dx = \frac{w}{3}\left(f(a_0) + 4f(a_1) + 2f(a_2) + 4f(a_3) + 2f(a_4) + \cdots + f(a_n)\right)$$

(b) Use Simpson's rule with $n = 6$ to evaluate an approximation for $\int_0^3 \frac{2x+1}{x+1}\,dx$, correct to four decimal places. (3 marks)

$$\approx \frac{0.5}{3}\left(1 + 4(1.3333 + 1.6 + 1.7143) + 2(1.5 + 1.6667) + 1.75\right)$$

✓ ✓ ✓ endpts.

$= 4.6123$ ✓ (4 dp)

Note: should use graph + table for graph + y-values.

(c) Determine the exact value of $\int_0^3 \frac{2x+1}{x+1}\,dx$ and hence calculate the percentage error of the approximation from (b). (2 marks)

$= \ln\frac{1}{4} + 6$ or $6 - 2\ln 2 = 4.6137$

% error is 0.030% (2sf)

Question 14 **(7 marks)**

(a) The equation of a sphere with centre at $(2, -3, 1)$ is $x^2 + y^2 + z^2 = ax + by + cz - 2$.

Determine the values of a, b, c and the radius of the circle. (3 marks)

$(x-2)^2 + (y+3)^2 + (z-1)^2 = r^2$

$\Rightarrow x^2 + y^2 + z^2 = 4x - 6y + 2z + 2 - 4 - 9 - 1 + r^2$ ✓

$\Rightarrow a = 4$

$b = -6$ ✓ and $2 - 14 + r^2 = 0$

$c = 2$

$r^2 = 12$

$r = \sqrt{12} = 2\sqrt{3}$ units ✓

(b) Two particles, P and Q, leave their initial positions at the same time and travel with constant velocities shown in the table below.

Particle	Initial position	Velocity
P	$10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$	$6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$
Q	$28\mathbf{i} + 22\mathbf{j} - 31\mathbf{k}$	$2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

Show that the two particles collide, stating the position vector of the point of collision. (4 marks)

$x_p = (10+6t)i + (-5+2t)j + (5-4t)k$

✓

$x_q = (28+2t)i + (22-4t)j + (-31+4t)k$

Collide if all components are equal at the same time (t value) ✓

$x: \quad 10 + 6t = 28 + 2t \Rightarrow t = \frac{18}{4} = 4.5$

$y: \quad -5 + 2t = 22 - 4t \Rightarrow t = \frac{27}{6} = 4.5$ ✓

$z: \quad 5 - 4t = -31 + 4t \Rightarrow t = \frac{36}{8} = 4.5$

$\therefore$ collide when $t = 4.5$ at $37i + 4j - 13k$ ✓

Question 15 **(8 marks)**

(a) Briefly describe a reason that a sample rather than a complete population may be used when carrying out a statistical investigation. (1 mark)

More efficient use of resources ✓

(Cheaper, quicker, can actually be done etc)

(b) A researcher used government records to select a random sample of the ages of 114 men who had died recently in a town close to an industrial complex. The mean and standard deviation of the ages in the sample were 73.3 and 8.27 years respectively.

(i) Explain why the sample standard deviation is a reasonable estimate for the population standard deviation in this case. (1 mark)

✓ Large enough sample of same distn.

(Should use s_{n-1})

(ii) Calculate a 98% confidence interval for the population mean and explain what the interval shows. (4 marks)

$$73 \pm \frac{z_{98} \times 8.27}{\sqrt{114}} \checkmark \qquad z_{98} = 2.326 \checkmark$$

$$\Rightarrow 71.5 \leq \mu \leq 75.1 \checkmark$$

98% confident that the mean falls in this interval ✓

(iii) The national average life-span of men was known to be 75.3 years. State with a reason what conclusion the researcher could draw from the confidence interval calculated in (ii) about the life-span of men in the town. (2 marks)

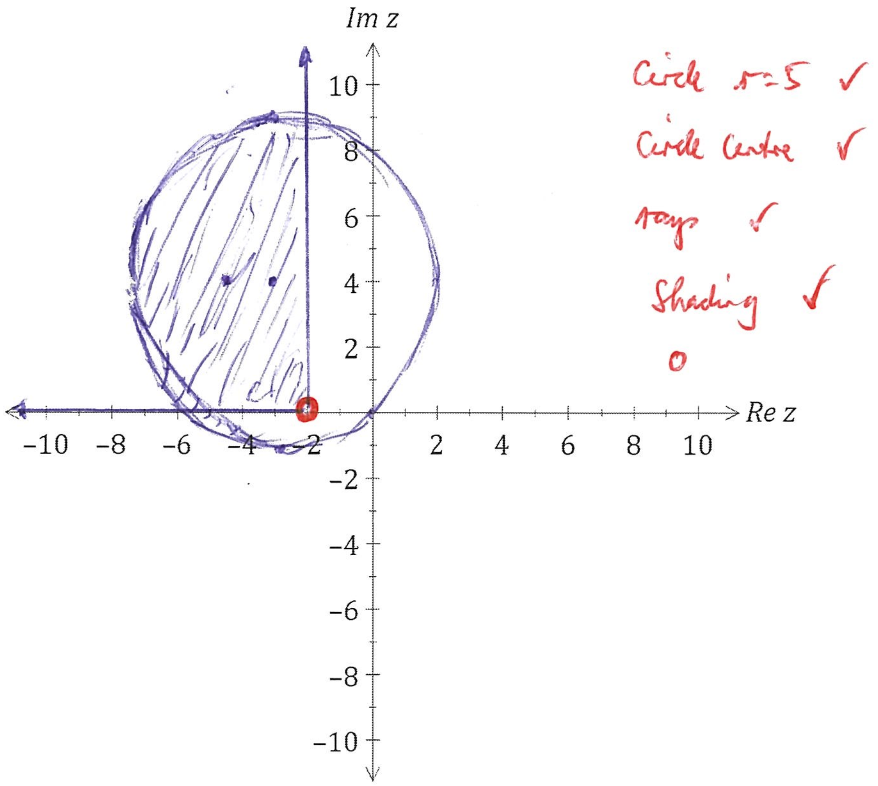
$\mu = 75.3$ is not within the C.I. ✓

Results are not consistent

i.e. sample differs from $\mu = 75.3$ ✓

Question 16 **(8 marks)**

(a) On the Argand diagram below, clearly show the region that satisfies the complex inequalities given by $|z + 3 - 4i| \leq 5$ and $\frac{\pi}{2} \leq \arg(z + 2) \leq \pi$. (4 marks)



(b) Determine all roots of the equation $z^5 = 16\sqrt{3} + 16i$, expressing them in the form $r \text{ cis } \theta$, where $r \geq 0$ and $-\pi \leq \theta \leq \pi$. (4 marks)

$z^5 = 32 \text{ cis } \frac{\pi}{6}$ ✓

$\therefore$ $z_1 = 2 \text{ cis } \frac{\pi}{30}$ ✓

$z_2 = 2 \text{ cis } \left(\frac{\pi}{30} + \frac{2\pi}{5}\right) = 2 \text{ cis } \frac{13\pi}{30}$

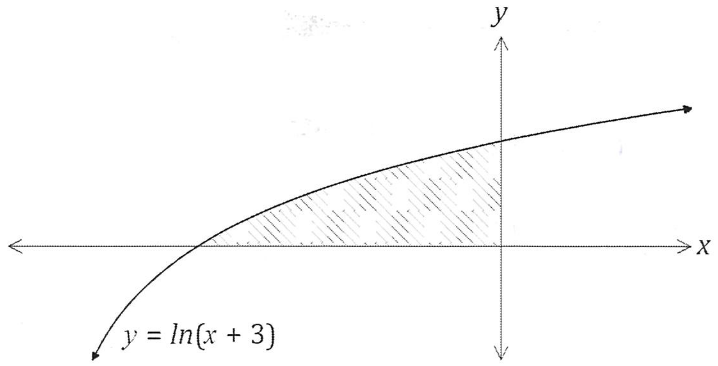
$z_3 = 2 \text{ cis } \left(\frac{\pi}{30} + \frac{4\pi}{5}\right) = 2 \text{ cis } \frac{25\pi}{30} = 2 \text{ cis } \frac{5\pi}{6}$ ✓

$z_4 = 2 \text{ cis } \left(\frac{\pi}{30} - \frac{2\pi}{5}\right) = 2 \text{ cis} \left(-\frac{11\pi}{30}\right)$

$z_5 = 2 \text{ cis } \left(\frac{\pi}{30} - \frac{4\pi}{5}\right) = 2 \text{ cis} \left(-\frac{23\pi}{30}\right)$ ✓

Question 17 **(7 marks)**

A region is bounded by $x = 0, y = 0$ and $y = \ln(x + 3)$ as shown in the graph below.



(a) Show analytically that the area of the region is given by $\int_0^{\ln 3} (3 - e^y)\, dy$. (3 marks)
(You do not need to evaluate this integral).

$x = 0 \Rightarrow y = \ln 3$

$y = \ln(x+3)$

$e^y = x + 3$

$x = e^y - 3$ ✓

$\text{Area} = -\int_c^d x\, dy$ ✓

$= \int_0^{\ln 3} 3 - e^y\, dy$

✓ limits

(b) Determine the exact volume of the solid generated when the region is rotated through 2π about the y-axis. (~~4~~ marks)
5 Stct.

$V_y = \pi \int_c^d x^2\, dy$ ✓

$= \pi \int_0^{\ln 3} (e^y - 3)^2\, dy$ ✓

$= \pi \int_0^{\ln 3} e^{2y} - 6e^y + 9$ or use ClassPad (5.93)

$= \pi \left(\frac{e^{2y}}{2} - 6e^y + 9y \right) \Big|_0^{\ln 3}$ ✓

$= \pi \left(\frac{9}{2} - 18 + 9\ln 3 \right) - \pi \left(\frac{1}{2} - 6 + 0 \right)$

$= 9\pi \ln 3 - 8\pi$ units^3 ✓

Note: $2\pi \int_{-2}^{0} x \ln(x+3)\, dx$ gives the negative answer (because of pos^n of region)
Could use $|\ |$s.

Question 18 **(8 marks)**

(a) A small object has initial position vector $\mathbf{r}(0) = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$ metres and moves with velocity vector given by $\mathbf{v}(t) = 2t\mathbf{i} - 4t\mathbf{j} + 3\mathbf{k}$ ms^{-1}, where t is the time in seconds.

(i) Show that the acceleration of the object is constant and state the magnitude of the acceleration. (2 marks)

$\vec{a}(t) = 2i - 4j$ ✓ a constant

$|\vec{a}(t)| = \sqrt{20} =$ ✓ $2\sqrt{5}$ m sec^{-1}

(ii) Determine the position vector of the object after 2 seconds. (3 marks)

$\vec{r}(t) = (t^2+1)i + (-2t^2+3)j + (3t-1)k$ ✓ variables ✓ constants

$\therefore \vec{r}(2) = 5i - 5j + 5k$ ✓

(b) Another small object has position vector given by $\mathbf{r}(t) = (1 + 2\sec t)\mathbf{i} + (3\tan t - 2)\mathbf{j}$ m, where t is the time in seconds.

Use a suitable trigonometric identity to derive the Cartesian equation of the path of this object. (3 marks)

$x = 1 + 2\sec t$

$y = 3\tan t - 2$ ✓

$\tan^2 t + 1 = \sec^2 t \Rightarrow \left(\frac{y+2}{3}\right)^2 + 1 = \left(\frac{x-1}{2}\right)^2$ ✓

~~vAAt~~

$\therefore \left(\frac{x-1}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$ ✓ or similar

$\frac{x^2-2x+1}{4} - \frac{y^2+4y+4}{9} = 1$; $9x^2 - 4y^2 - 18x - 16y = 8$

Question 19 **(5 marks)**

Apple believes that 60% of mobile phone users will eventually buy an iPhone 7. Initial sales were 2% of the total market, rising to 7% after 3 weeks.

(a) Use the logistic model to predict the total sales after 7 weeks. (3 marks)

$$P = \frac{m.P_0}{P_0 + (m - P_0)e^{-kmt}}$$ ✓

$P_0 = 2 \quad m = 60 \quad P = 7$ at $t = 3$

$\Rightarrow k = 7.461 \times 10^{-3}$ ✓

$\Rightarrow P(7) = 26.5\%$ ✓

(b) This logistic model is based on the differential equation $\frac{dN}{dt} = aN - bN^2$.

x Evaulate a and b. (2 marks)

$b = k = 7.461 \times 10^{-3}$ ✓

$\frac{a}{b} = m \quad \Rightarrow a = 0.4476$ ✓

Question 20 **(7 marks)**

(a) A particle undergoing simple harmonic motion with a period of 5 seconds is observed to move in a straight line, oscillating 3.6 m either side of a central position. Determine the speed of the particle when it is 3 m from the central position. (3 marks)

$A = 3.6$ $\quad \frac{2\pi}{k} = 5 \Rightarrow k = \frac{2\pi}{5}$ ✓

$v^2 = k^2(A^2 - x^2)$

$= \frac{4\pi}{25}(3.6^2 - 9)$ ✓

$\therefore |v| = \frac{2\pi\sqrt{99}}{25}$ or 2.50 m/sec ✓

(b) Another particle moving in a straight line experiences an acceleration of $x + 2.5$ ms^{-2}, where x is the position of the particle at time t seconds.

Given that when $x = 1$, the particle had a velocity of 2 ms^{-1}, determine the velocity of the particle when $x = 2$. (4 marks)

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = x + 2.5$ ✓

$\therefore \frac{1}{2}v^2 = \frac{x^2}{2} + 2.5x + C$ ✓

$x = 1, v = 2 \Rightarrow 2 = \frac{1}{2} + 2.5 + C$

$C = -1$

$\therefore v^2 = x^2 + 5x \cancel{+} - 2$ ✓

$v^2(x = 2) = 12$

$\therefore v(x = 2) = \cancel{\pm} 2\sqrt{3}$ m.sec^{-1} ✓

$(v(1) > 0,\ a > 0,\ \text{so } v(2) > 0)$

Note: $a = \frac{dv}{dt} = x + 2.5 \Rightarrow \frac{dv}{dx} \cdot \frac{dx}{dt} = x + 2.5 \Rightarrow \int v \cdot dv = \int x + 2.5\, dx$

since $\frac{dx}{dt} = v$

Question 21 **(8 marks)**

The complex numbers w and z are given by $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ and $r(\cos\theta + i\sin\theta)$ respectively, where $r > 0$ and $-\frac{\pi}{3} < \theta < \frac{\pi}{3}$.

(a) State, in terms of r and θ, the modulus and argument of wz and $\frac{z}{w}$. (3 marks)

$w = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$= \text{cis}\left(-\frac{2\pi}{3}\right)$ ✓

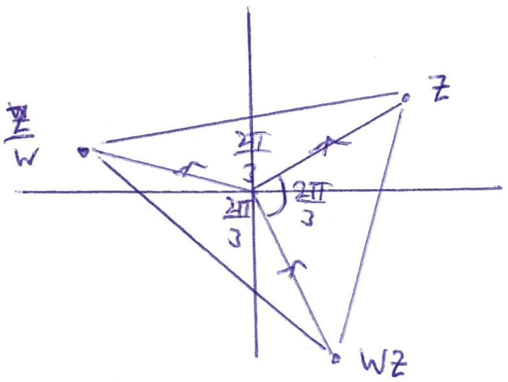
$|wz| = r$

$\arg(wz) = \theta - \frac{2\pi}{3}$ ✓

$\left|\frac{z}{w}\right| = r$

$\arg\left(\frac{z}{w}\right) = \theta + \frac{2\pi}{3}$ ✓

(b) Explain why the points represented by z, wz and $\frac{z}{w}$ in an Argand diagram are the vertices of an equilateral triangle. (2 marks)



Triangle divides into 3 congruent isosceles Δ's, sides ✓ r, central angle $\frac{2\pi}{3}$ ✓

∴ large Δ is equilateral

(c) In an Argand diagram, one of the vertices of an equilateral triangle is represented by the complex number $5-\sqrt{3}i$. If the other two vertices lie on a circle with centre at the origin, determine the complex numbers they represent in exact Cartesian form. (3 marks)

Let $z = 5 - \sqrt{3}i$ ✓

$\therefore wz = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)(5 - \sqrt{3}i)$

$= -2.5 + \frac{\sqrt{3}}{2}i - \frac{5\sqrt{3}}{2}i - \frac{3}{2}$

$= -4 - 2\sqrt{3}i$ ✓

& $\frac{z}{w} = \frac{5-\sqrt{3}i}{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)} = -1 + 3\sqrt{3}i$ (ClassPad)

or $= \frac{-2.5 + \frac{\sqrt{3}}{2}i + \frac{5\sqrt{3}}{2}i + \frac{3}{2}}{\frac{1}{4} + \frac{3}{4} = 1}$

$= -1 + 3\sqrt{3}i$ ✓

End of questions

Additional working space

Question number: ___________